SIX-DIMENSIONAL EXCEPTIONAL QUOTIENT SINGULARITIES

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Abstract. We classify six-dimensional exceptional quotient singularities and show that seven-dimensional exceptional quotient singularities do not exist. Inter alia we prove that the irreducible six-dimensional projective representation of the sporadic simple Hall–Janko group gives rise to an exceptional quotient singularity.

We assume that all varieties are projective, normal, and defined over $\mathbb{C}$.

1. Introduction

Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity (see [13, Definition 3.5]), and let $\xi\colon \bar{V} \to V$ be a resolution of singularities of the variety V. Then

$$K_{\bar{V}} \sim_{\mathbb{Q}} \xi^*\big(K_V\big) + \sum_{i=1}^{r} b_i E_i,$$

where E_i is a ξ-exceptional divisor, and $b_i \in \mathbb{Q}$. Let B be an effective $\mathbb{Q}$-divisor on V. Put

$$B = \sum_{i=1}^{m} a_i B_i,$$

where B_i is a prime Weil divisor on V, and $a_i \in \mathbb{Q}_{\geqslant 0}$. Suppose that B is a $\mathbb{Q}$-Cartier divisor. Then

$$\sum_{i=1}^{m} a_i \bar{B}_i \sim_{\mathbb{Q}} \xi^*\left(\sum_{i=1}^{m} a_i B_i\right) - \sum_{i=1}^{r} c_i E_i,$$

where $\bar{B}_i$ is the proper transform of the divisor B_i on the variety $\bar{V}$. Suppose that

$$\left(\bigcup_{i=1}^{m} \bar{B}_i\right) \bigcup \left(\bigcup_{i=1}^{r} E_i\right)$$

is a divisor with simple normal crossing.

Definition 1.1 ([13, Definition 8.1])**.** The log canonical threshold of the divisor B at O is

$$\mathrm{c}_O\big(X, B\big) = \min\left(\min\left\{\frac{1}{a_i}\,\middle|\, O \in B_i\right\}, \min\left\{\frac{b_i+1}{c_i}\,\middle|\, O \in \xi\big(E_i\big)\right\}\right) \in \mathbb{Q}_{>0} \cup \big\{+\infty\big\}$$

Definition 1.2 ([8, Definition 2.3.1])**.** The log canonical multiplicity of the divisor B at O is

$$\mu_O\big(X, B\big) = \max\left\{\alpha + \beta \,\middle|\, \begin{array}{l} O \in \xi\left(\left(\bar{B}_{i_1} \cap \ldots \cap \bar{B}_{i_\alpha}\right) \bigcap \left(E_{k_1} \cap \ldots \cap E_{k_\beta}\right)\right) \text{ and} \\ \dfrac{1}{a_{i_1}} = \ldots = \dfrac{1}{a_{i_\alpha}} = \dfrac{b_{k_1}+1}{c_{k_1}} = \ldots = \dfrac{b_{k_\beta}+1}{c_{k_\beta}} = \mathrm{c}_O\big(X, B\big) \end{array}\right\} \in \mathbb{Z}_{\geqslant 0},$$

The authors were partially supported by AG Laboratory GU-HSE, RF government grant 11 11.G34.31.0023. The first author was supported by the grants NSF DMS-0701465 and EPSRC EP/E048412/1, the second author was supported by the grants RFFI 08-01-00395-a, RFFI 11-01-00185-a, RFFI 11-01-00336-a, N.Sh.-1987.2008.1 and N.Sh.-4713.2010.1.

One can show that t he numbers $c_O(X,B)$ and $\mu_O(X,B)$ do not depend on the choice of the log resolution ξ.

Definition 1.3 ([20, Definition 2.5])**.** The singularity $(V \ni O)$ is exceptional if

$$\mu_O\big(X,B\big) \leqslant 1$$

for every effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor B on the variety V.

Note that Definition 1.3 looks different from [20, Definition 2.5], but they are equivalent. One can show that exceptional singularities exist in any dimension greater than 1 (see [7, Example 3.13]).

Example 1.4. Suppose that $\dim(V)=2$ and $-K_V$ is Cartier. Then the singularity $(V \ni O)$ is exceptional if and only if it is a Du Val singularity of type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$.

Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$, where $n \geqslant 1$. Put

$$\bar{G} = \phi\big(G\big) \subset \mathrm{Aut}\big(\mathbb{P}^n\big) \cong \mathrm{PGL}_{n+1}\Big(\mathbb{C}\Big),$$

where $\phi\colon \mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^n)$ is the natural projection. Put

$$\mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (\mathbb{P}^n, \lambda D) \text{ has log canonical singularities}\\ \text{for every } \bar{G}\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}\end{array}\right\} \in \mathbb{R}.$$

Remark 1.5 (cf. Appendix A)*.* It follows from [6, Theorem A.3] that

$$\mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) = \alpha_{\bar{G}}\big(\mathbb{P}^n\big),$$

where $\alpha_{\bar{G}}(\mathbb{P}^n)$ is the $\bar{G}$-invariant α-invariant introduced in [30] and [31].

We are going to study the quotient singularity $\mathbb{C}^{n+1}/G$.

Remark 1.6*.* Let $R \subseteq G$ be a subgroup generated by all reflections in G (see [28, §4.1]). Then the quotient $\mathbb{C}^{n+1}/R$ is isomorphic to $\mathbb{C}^{n+1}$ (see [28, Theorem 4.2.5]). Moreover, the subgroup $R \subseteq G$ is normal, and the singularity $\mathbb{C}^{n+1}/G$ is isomorphic to the singularity $\mathbb{C}^{n+1}/(G/R)$. Note that the subgroup R is trivial if $G \subset \mathrm{SL}_{n+1}(\mathbb{C})$. If $G = R$ (in particular, if G is a trivial group), then the singularity $\mathbb{C}^{n+1}/G \cong \mathbb{C}^{n+1}$ is not exceptional.

The following result provides a characterization of exceptional quotient singularities.

Theorem 1.7 ([7, Theorem 3.17])**.** Let $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ be a finite subgroup that does not contain reflections. Then $\mathbb{C}^{n+1}/G$ is exceptional if and only if for any $\bar{G}$-invariant effective $\mathbb{Q}$-divisor D on $\mathbb{P}^n$ such that $D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}$, the log pair $(\mathbb{P}^n, D)$ is Kawamata log terminal.

Corollary 1.8. Let $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ be a finite subgroup that does not contain reflections. Then the singularity $\mathbb{C}^{n+1}/G$ is exceptional if $\mathrm{lct}(\mathbb{P}^n, \bar{G}) > 1$. Moreover, the singularity $\mathbb{C}^{n+1}/G$ is not exceptional if either $\mathrm{lct}(\mathbb{P}^n, \bar{G}) < 1$, or G has a semi-invariant of degree at most $n+1$.

Note that the assumption that G contains no reflections is crucial for Theorem 1.7.

Example 1.9. Let $G \subset \mathrm{GL}_4(\mathbb{C})$ be the subgroup #32 in [26, Table VII]. Then G is generated by reflections (see [26]). Thus, the singularity $\mathbb{C}^4/G$ is not exceptional by Remark 1.6. On the other hand, it follows from [7, Theorem 4.13] that $\mathrm{lct}(\mathbb{P}^3, \bar{G}) \geqslant 5/4$. One can produce similar examples for two-dimensional and three-dimensional singularities.

Definition 1.10 (see [3])**.** The subgroup $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is primitive if there is no non-trivial decomposition

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^{r} V_i$$

such that for any $g \in G$ and any i there is some $j = j(g)$ such that $g(V_i) = V_j$. The subgroup $\bar{G} \subset \mathrm{PGL}_{n+1}(\mathbb{C})$ is primitive if G is primitive.

Up to conjugation, there are finitely many primitive finite subgroups in $\mathrm{SL}_{n+1}(\mathbb{C})$ (see [9]).

Theorem 1.11 (see [24, Proposition 2.1] or [7, Corollary 3.20])**.** Let $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ be a finite subgroup that does not contain reflections. If $\mathbb{C}^{n+1}/G$ is exceptional, then G is primitive.

Exceptional quotient singularities of dimension up to 5 are classified in [27], [20], [7].

Theorem 1.12 ([27], [20], [7, Theorem 1.22])**.** Let $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ be a finite subgroup without reflections. If $n \leqslant 4$, then the following are equivalent:

- the singularity $\mathbb{C}^{n+1}/G$ is exceptional,
- the inequality $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \geqslant (n+2)/(n+1)$ holds,
- the group G is primitive and has no semi-invariants of degree at most $n+1$.

The assertion of Theorem 1.12 is no longer true if $n \geqslant 5$ (see [7, Example 3.25]).

Remark 1.13. Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be a finite subgroup. Then

$$\mathrm{lct}\Big(\mathbb{P}^1, \bar{G}\Big) = \begin{cases} 6 \text{ if } \bar{G} \cong \mathrm{A}_5, \\ 4 \text{ if } \bar{G} \cong \mathrm{S}_4, \\ 2 \text{ if } \bar{G} \cong \mathrm{A}_4, \\ 1 \text{ if } \bar{G} \cong \mathrm{D}_m, \\ 1/2 \text{ if } \bar{G} \cong \mathbb{Z}_m. \end{cases}$$

The main purpose of this paper is to prove the following result.

Theorem 1.14. Let $G \subset \mathrm{SL}_6(\mathbb{C})$ be a finite subgroup. Then the following are equivalent:

- the singularity $\mathbb{C}^6/G$ is exceptional,
- the inequality $\mathrm{lct}(\mathbb{P}^5, \bar{G}) \geqslant 7/6$ holds,
- either $\bar{G}$ is the Hall–Janko group HaJ (see [16], [17]), or $G \cong 6.\mathrm{A}_7$ and $\bar{G} \cong \mathrm{A}_7$.

Proof. The required assertion follows from Theorems 3.3, 4.2, 4.4 and Lemma 2.9. □

As far as we know, Theorem 1.14 gives the first appearance of the Hall–Janko group HaJ in algebraic geometry. The assertion of Theorem 1.14 gives new examples of normalized Kähler–Ricci iterations that converge to the Fubini–Study metric on $\mathbb{P}^5$ (see [25], cf. [7, Question 1.9]). Furthermore, it follows from Corollary 1.8 that Theorem 1.14 can be considered as a classification of six-dimensional exceptional quotient singularities.

Remark 1.15. Suppose that $\bar{G} \subset \mathrm{PGL}_6(\mathbb{C})$. If $\bar{G} \cong \mathrm{HaJ}$, then there exist two subgroups in $\mathrm{SL}_6(\mathbb{C})$ whose images in $\mathrm{PGL}_6(\mathbb{C})$ coincide with $\bar{G}$. Namely, one of them is isomorphic to 2.HaJ, and another one is isomorphic to the extension of the subgroup $2.\mathrm{HaJ} \subset \mathrm{SL}_6(\mathbb{C})$ by a scalar matrix with non-zero entries equal to a primitive root of unity of degree 6. Moreover, up to conjugation $\mathrm{PGL}_6(\mathbb{C})$ contains a unique subgroup isomorphic to HaJ. On the other hand, $\mathrm{PGL}_6(\mathbb{C})$ contains non-conjugate subgroups isomorphic to $\mathbb{A}_7$. Furthermore, if $\bar{G} \cong \mathbb{A}_7$ and the singularity $\mathbb{C}^6/G$ is exceptional, then we must necessarily have $G \cong 6.\mathrm{A}_7$, which uniquely determines the subgroup $\bar{G} \subset \mathrm{PGL}_6(\mathbb{C})$ up to conjugation. Other alternatives for a minimal lift G of a primitive group $\mathrm{A}_7 \cong \bar{G} \subset \mathrm{PGL}_6(\mathbb{C})$ are $G \cong 3.\mathrm{A}_7$ and $G \cong \mathrm{A}_7$ (see Theorem 3.2), which happens for two other classes of subgroups $\mathrm{A}_7 \cong \bar{G} \subset \mathrm{PGL}_6(\mathbb{C})$. In the latter cases the singularity $\mathbb{C}^6/G$ is not exceptional.

Finally, we prove the following surprising result.

Theorem 1.16 (cf. [7, Example 3.13])**.** *There are no exceptional quotient singularities of dimension 7.*

The plan of the paper is as follows. In Section 2 we collect well known auxiliary results. In Section 3 we show that apart from the singularities related to the groups $6.\mathrm{A}_7$ and 2.HaJ all other six-dimensional quotient singularities are not exceptional. In Section 4 we prove the exceptionality of the singularities related to the groups $6.\mathrm{A}_7$ and 2.HaJ thus completing the proof of Theorem 1.14. Finally, in Section 5 we prove Theorem 1.16. In Appendix A we introduce a new invariant of a Kawamata log terminal singularity based on the classical α-invariant of Tian.

Throughout the paper we use usual notation for cyclic, dihedral, symmetric and alternating groups, as well as for standard algebraic groups. For a group Γ we denote by $k.\Gamma$ a (non-trivial) central extension of Γ by the central subgroup $\mathbb{Z}_k$ (this might be non-unique).

Many of the computations with the characters of large finite groups we need are too complicated to make by hand (namely, those mentioned in the proofs of Theorem 3.3, 4.2 and 4.4 and in Remark 3.4). In such cases we used the Magma software [2].

The authors would like to thank G. Robinson for numerous useful explanations and comments, and T. Dokchitser and A. Khoroshkin for computational support.

2. Preliminaries

Let X be a variety with at most Kawamata log terminal singularities (see [13, Definition 3.5]), let B_X be an effective $\mathbb{Q}$-divisor on the variety X such that (X, B_X) is log canonical. Then

$$B_X = \sum_{i=1}^{r} a_i B_i,$$

where $a_i \in \mathbb{Q}_{\geqslant 0}$, and B_i is a prime Weil divisor on the variety X.

Let $\pi\colon \bar{X} \to X$ be a birational morphism such that $\bar{X}$ is smooth. Then

$$K_{\bar{X}} + \sum_{i=1}^{r} a_i \bar{B}_i \sim_{\mathbb{Q}} \pi^*\Big(K_X + B_X\Big) + \sum_{i=1}^{m} d_i E_i,$$

where $\bar{B}_i$ is the proper transforms of the divisor B_i on the variety $\bar{X}$, and E_i is an exceptional divisor of the morphism π, and d_i is a rational number. We may assume that

$$\left(\bigcup_{i=1}^{r} \bar{B}_i\right) \bigcup \left(\bigcup_{i=1}^{m} E_i\right)$$

is a divisor with simple normal crossing. Put

$$\mathcal{I}\Big(X, B_X\Big) = \pi_* \mathcal{O}_{\bar{X}}\left(\sum_{i=1}^{m} \lceil d_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor B_i\right).$$

Theorem 2.1 ([15, Theorem 9.4.8])**.** *Let H be a nef and big $\mathbb{Q}$-divisor on X such that*

$$K_X + B_X + H \equiv D,$$

where D is a Cartier divisor on the variety X. Then $H^i(\mathcal{I}(X, B_X) \otimes D) = 0$ for every $i \geqslant 1$.

Let $\mathcal{L}(X, B_X)$ be a subscheme that corresponds to the ideal sheaf $\mathcal{I}(X, B_X)$. Put

$$\mathrm{LCS}\Big(X, B_X\Big) = \mathrm{Supp}\bigg(\mathcal{L}\Big(X, B_X\Big)\bigg).$$

The subscheme $\mathcal{L}(X, B_X)$ is reduced, because (X, B_X) is log canonical. Note that

- $\mathcal{I}(X, B_X)$ is known as the multiplier ideal sheaf (see [15, Section 9.2]),

- $\mathcal{L}(X, B_X)$ is known as the log canonical singularities subscheme (see [6, Definition 2.5]),
- $\mathrm{LCS}(X, B_X)$ is known as the locus of log canonical singularities (see [27, Definition 3.14]).

Let Z be a center of log canonical singularities of the log pair (X, B_X) (see [11, Definition 1.3]), and let $\mathbb{LCS}(X, B_X)$ be the set of all centers of log canonical singularities of the log pair (X, B_X).

Lemma 2.2 ([11, Proposition 1.5])**.** Let Z' be an element of the set $\mathbb{LCS}(X, B_X)$ such that

$$\varnothing \neq Z \cap Z' = \sum_{i=1}^{k} Z_i,$$

where $Z_i \subsetneq Z$ is an irreducible subvariety. Then $Z_i \in \mathbb{LCS}(X, B_X)$ for every $i \in \{1, \ldots, k\}$.

Suppose that Z is a minimal center in $\mathbb{LCS}(X, B_X)$ (see [11], [12]).

Theorem 2.3 ([12, Theorem 1])**.** The variety Z is normal and has at most rational singularities. For every ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor Δ on X there exists an effective $\mathbb{Q}$-divisor B_Z on the variety Z such that

$$\Big(K_X + B_X + \Delta\Big)\Big|_Z \sim_{\mathbb{Q}} K_Z + B_Z,$$

and (Z, B_Z) has Kawamata log terminal singularities.

Remark 2.4. In the notation and assumptions of Theorem 2.3, suppose that $K_X + B_X + \Delta \sim_{\mathbb{Q}} D$, where D is a Cartier divisor on X. Put $H = D|_Z$. Let $\nu\colon \bar{Z} \to Z$ be a desingularization. Then

$$h^0\Big(\mathcal{O}_Z\big(H\big)\Big) = \chi\Big(\mathcal{O}_Z\big(H\big)\Big) = \chi\bigg(\mathcal{O}_{\bar{Z}}\Big(\nu^*\big(H\big)\Big)\bigg)$$

by Theorem 2.1, because Z has at most rational singularities by Theorem 2.3.

Let $\bar{G} \subseteq \mathrm{Aut}(X)$ be a finite subgroup. Suppose that B_X is $\bar{G}$-invariant. Then $g(Z) \in \mathbb{LCS}(X, B_X)$ for every $g \in \bar{G}$, and the locus $\mathrm{LCS}(X, B_X)$ is $\bar{G}$-invariant. It follows from Lemma 2.2 that

$$g\big(Z\big) \cap g'\big(Z\big) \neq \varnothing \iff g\big(Z\big) = g'\big(Z\big)$$

for every $g \in \bar{G} \ni g'$, because Z is a minimal center in $\mathbb{LCS}(X, B_X)$.

Lemma 2.5. Suppose that the divisor B_X is ample. Let ϵ be an arbitrary rational number such that $\epsilon > 1$. Then there exists an effective $\bar{G}$-invariant $\mathbb{Q}$-divisor D on the variety X such that

$$\mathbb{LCS}\Big(X, D\Big) = \bigcup_{g \in \bar{G}} \Big\{g\big(Z\big)\Big\},$$

the log pair (X, D) is log canonical, and the equivalence $D \sim_{\mathbb{Q}} \epsilon(B_X)$ holds.

Proof. See the proofs of [11, Theorem 1.10], [12, Theorem 1], [7, Lemma 2.8]. □

Suppose that $X \cong \mathbb{P}^n$. Let H be a hyperplane in $\mathbb{P}^n$. Suppose that

$$\mathbb{LCS}\Big(X, B_X\Big) = \bigcup_{g \in \bar{G}} \Big\{g\big(Z\big)\Big\},$$

and let Y be the $\bar{G}$-orbit of the subvariety $Z \subset \mathbb{P}^n$.

Lemma 2.6. Put $s = n - \dim(Y)$ and

$$r = \begin{cases} \lceil \mu - s - 1 \rceil + 1 \text{ if } \mu \in \mathbb{Z}, \\ \lceil \mu - s - 1 \rceil \text{ if } \mu \notin \mathbb{Z}, \end{cases}$$

where $\mu \in \mathbb{Q}$ such that $B_X \sim_{\mathbb{Q}} \mu H$. Then $r \geqslant 0$ and

$$\deg\big(Y\big) \leqslant \binom{s + r}{r}.$$

Proof. Let $\Pi\subset\mathbb{P}^n$ be a general linear subspace of dimension s. Put

$$D=B_X\Big|_{\Pi}$$

and $\Lambda=H\cap\Pi$. Then $\deg(Y)=|Y\cap\Pi|$ and $\mathrm{LCS}(\Pi,D)=Y\cap\Pi$. One has

$$K_{\Pi}+D\sim_{\mathbb{Q}}\Big(\mu-s-1\Big)\Lambda.$$

It follows from Theorem 2.1 that there is an exact sequence of cohomology groups

$$0\longrightarrow H^0\Bigg(\mathcal{O}_{\Pi}\big(r\Lambda\big)\otimes\mathcal{I}\Big(\Pi,D\Big)\Bigg)\longrightarrow H^0\Big(\mathcal{O}_{\Pi}\big(r\Lambda\big)\Big)\longrightarrow H^0\Big(\mathcal{O}_{\mathcal{L}(\Pi,D)}\Big)\longrightarrow 0,$$

and $\mathrm{Supp}(\mathcal{L}(\Pi,D))=\mathrm{LCS}(\Pi,D)=Y\cap\Pi\neq\varnothing$. Therefore, we see that $r\geqslant 0$ and

$$\deg\big(Y\big)=\big|Y\cap\Pi\big|\leqslant h^0\Big(\mathcal{O}_{\mathcal{L}(\Pi,D)}\Big)\leqslant h^0\Big(\mathcal{O}_{\Pi}\big(r\Lambda\big)\Big)=h^0\Big(\mathcal{O}_{\mathbb{P}^s}\big(r\big)\Big)=\binom{s+r}{r},$$

which completes the proof. □

Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ such that $\bar{G}=\phi(G)$, where $\phi\colon\mathrm{GL}_{n+1}(\mathbb{C})\to\mathrm{Aut}(\mathbb{P}^n)\cong\mathrm{PGL}_{n+1}(\mathbb{C})$ is the natural projection.

Lemma 2.7. If G is conjugate to a subgroup in $\mathrm{GL}_{n+1}(\mathbb{R})$, then G has an invariant of degree 2.

Proof. If G is conjugate to a subgroup in $\mathrm{GL}_{n+1}(\mathbb{R})$, then there exists a (real positive definite) G-invariant inner product, which gives a non-trivial G-invariant element in $\mathrm{Sym}^2(\mathbb{C}^{n+1})$. □

Lemma 2.8. Suppose that there exists a normal subgroup $F\subset G$ such that G/F is abelian, and F has an invariant of degree d. Then G has a semi-invariant of degree d.

Proof. Let V be a space of invariants of the group F of degree d. Then the group G/F naturally acts on the space V. Since the group G/F is abelian, it has a one-dimensional invariant subspace, which gives a required semi-invariant of the subgroup G. □

Let $G_1\subset\mathrm{SL}_2(\mathbb{C})$ and $G_2\subset\mathrm{SL}_l(\mathbb{C})$ be finite subgroups, let $\mathbb{M}$ be the vector space of $2\times l$-matrices with entries in $\mathbb{C}$. For every $(g_1,g_2)\in G_1\times G_2$ and every $M\in\mathbb{M}$, put

$$\Big(g_1,g_2\Big)\big(M\big)=g_1Mg_2^{-1}\in\mathbb{M}\cong\mathbb{C}^{2l},$$

which induces a homomorphism $\varphi\colon G_1\times G_2\to\mathrm{SL}_{2l}(\mathbb{C})$. Note that $|\ker(\varphi)|\leqslant 2$ if n is even, and φ is a monomorphism if n is odd. Suppose that $n=2l-1\geqslant 3$.

Lemma 2.9 ([7, Lemma 3.24])**.** Suppose that $G=\varphi(G_1\times G_2)$. Then $\mathrm{lct}(\mathbb{P}^n,\bar{G})<1$.

Proof. Put $s=l-1$. Let $\psi\colon\mathbb{P}^1\times\mathbb{P}^s\to\mathbb{P}^n$ be the Segre embedding. Put $Y=\psi(\mathbb{P}^1\times\mathbb{P}^s)$ and let $\mathcal{Q}$ be the linear system consisting of all quadric hypersurfaces in $\mathbb{P}^n$ that pass through the subvariety Y. Then $\mathcal{Q}$ is a non-empty $\bar{G}$-invariant linear system. The log pair $(\mathbb{P}^n,l\mathcal{Q})$ is not log-canonical along Y. Now it follows from [13, Theorem 4.8] that $\mathrm{lct}(\mathbb{P}^n,\bar{G})<1$. □

3. Six-dimensional case

Let G be a finite subgroup in $\mathrm{SL}_{n+1}(\mathbb{C})$. Put $V=\mathbb{C}^{n+1}$.

Definition 3.1 (see [19, §1])**.** The subgroup G is quasiprimitive if the following conditions hold:

- the vector space V is an irreducible representation of the group G,
- for any nontrivial normal subgroup $N\subseteq G$ one has $V\cong W^{\oplus r}$ as a representation of N, where W is an irreducible representation of N, and $r\geqslant 1$.

Suppose that $n=5$. Let $\phi\colon\mathrm{SL}_6(\mathbb{C})\to\mathrm{Aut}(\mathbb{P}^5)$ be the natural projection. Put $\bar{G}=\phi(G)$. We say that the subgroup G is the lift of the subgroup $\bar{G}\subset\mathrm{Aut}(\mathbb{P}^5)\cong\mathrm{PGL}_6(\mathbb{C})$ to $\mathrm{SL}_6(\mathbb{C})$.

Theorem 3.2 ([18, §3]). Suppose that G is quasiprimitive. Then there exists a lift of the subgroup $\bar{G} \subset \mathrm{Aut}(\mathbb{P}^5)$ to $\mathrm{SL}_6(\mathbb{C})$ that is contained in the following list:

(I) (i) a subgroup of the group $\mathrm{SL}_6(\mathbb{C})$ that satisfies the hypotheses of Lemma 2.9,
(ii) a certain subgroup of a subgroup described in I(i) (see [18, §3] for details),
(II) $\mathrm{SL}_2(\mathbb{F}_5)$,
(III) $2.\mathrm{S}_5$,
(IV) (i) $3.\mathrm{A}_6$,
(ii) an extension of the subgroup described in IV(i) by an automorphism of order 2,
(V) $6.\mathrm{A}_6$,
(VI) A_7 or S_7,
(VII) $3.\mathrm{A}_7$,
(VIII) $6.\mathrm{A}_7$,
(IX) (i) $\mathrm{PSL}_2(\mathbb{F}_7)$,
(ii) $\mathrm{PGL}_2(\mathbb{F}_7)$,
(X) (i) $\mathrm{SL}_2(\mathbb{F}_7)$,
(ii) an extension of the subgroup described in X(i) by an automorphism of order 2,
(XI) $\mathrm{SL}_2(\mathbb{F}_{11})$,
(XII) $\mathrm{SL}_2(\mathbb{F}_{13})$,
(XIII) (i) $\mathrm{PSp}_4(\mathbb{F}_3)$,
(ii) an extension of the subgroup described in XIII(i) by an automorphism of order 2,
(XIV) (i) $\mathrm{SU}_3(\mathbb{F}_3)$,
(ii) an extension of the subgroup described in XIV(i) by an automorphism of order 2,
(XV) (i) $6.\mathrm{PSU}_4(\mathbb{F}_3)$,
(ii) an extension of the subgroup described in XV(i) by an automorphism of order 2,
(XVI) 2.HaJ, where HaJ is the Hall–Janko group (see [16], [17]),
(XVII) (i) $6.\mathrm{PSL}_3(\mathbb{F}_4)$,
(ii) an extension of the subgroup described in XVII(i) by an automorphism of order 2.

Recall that all primitive subgroups are quasiprimitive (see [19, §1]).

The main purpose of this section is to prove the following result.

Theorem 3.3. If G is primitive, then G has a semi-invariant of degree at most 6 unless there exists a lift of $\bar{G}$ to $\mathrm{SL}_6(\mathbb{C})$ that is a group of type I, VIII or XVI in the notation of Theorem 3.2.

Proof. Recall that changing a lift of $\bar{G}$ to $\mathrm{SL}_6(\mathbb{C})$ does not change the degrees of semi-invariants, which implies that we may assume that G is one of the groups listed in Theorem 3.2.

If the subgroup G is of type VI, IX(i) or XIII(i), then the subgroup G is conjugate to a subgroup of $\mathrm{SL}_6(\mathbb{Q})$ (see [10]), and hence G has an invariant of degree 2 by Lemma 2.7.

If the subgroup G is of type XV(i), then G is a subgroup of the Mitchell group $6.\mathrm{PSU}_4(\mathbb{F}_3).2$, which is the group #34 in [26, Table VII], and hence the subgroup G has an invariant of degree 6, because the Mitchell group has an invariant of degree 6 (see [26, Table VII]).

If the subgroup G is of type II, V, VII, X(i), XI, XII, XIV(i) or XVII(i), then the minimal degree d_{min} of the invariants of the subgroup G is given in the following table:

G	$2.\mathrm{A}_5$	$6.\mathrm{A}_6$	$3.\mathrm{A}_7$	$\mathrm{SL}_2(\mathbb{F}_7)$	$\mathrm{SL}_2(\mathbb{F}_{11})$	$\mathrm{SL}_2(\mathbb{F}_{13})$	$\mathrm{SU}_3(\mathbb{F}_3)$	$6.\mathrm{PSL}_3(\mathbb{F}_4)$
Type	II	V	VII	X(i)	XI	XII	XIV(i)	XVII(i)
d_{min}	4	6	3	4	4	4	6	6

If the subgroup G is a subgroup of type IV(i), then G is a subgroup of a quasiprimitive subgroup of type VII, which implies that the subgroup G has an invariant of degree 3.

If the subgroup G is a subgroup of type III, then it has a normal subgroup isomorphic to $2.\mathrm{A}_5$, which implies that G has a semi-invariant of degree 4 by Lemma 2.8.

Arguing as in the case of a subgroup of type III, we see that the subgroup G has a semi-invariant of degree 3, 2, 4, 2, 6, 6 or 6 in the case when the subgroup $G \subset \mathrm{SL}_6(\mathbb{C})$ is a quasiprimitive subgroup of type IV(ii), IX(ii), X(ii), XIII(ii), XIV(ii), XV(ii) or XVII(ii), respectively. □

Remark 3.4. In the notation of Theorem 3.2, if G is a primitive subgroup of type VIII or XVI, then a direct computation shows that the minimal degree of the semi-invariants of G equals 12.

4. Exceptional cases

Let G be a subgroup in $\mathrm{SL}_6(\mathbb{C})$. Define V and $\bar{G}$ as in Section 3.

Remark 4.1. If the group $\bar{G}$ is a simple non-abelian group such that $Z(G) \subseteq [G,G]$, where $Z(G)$ and $[G,G]$ denote the center and the commutator of the subgroup G, respectively, then every semi-invariant of the group G is its invariant.

Theorem 4.2. Suppose that $\bar{G} \cong \mathrm{HaJ}$ is the Hall–Janko group. Then $\mathrm{lct}(\mathbb{P}^5, \bar{G}) \geqslant 7/6$.

Proof. We may assume that $G \cong 2.\mathrm{HaJ}$ (see Theorem 3.2). Then $Z(G) \subseteq [G,G]$.

Suppose that $\mathrm{lct}(\mathbb{P}^5, \bar{G}) < 7/6$. Then there is an effective $\bar{G}$-invariant $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^5} \sim \mathcal{O}_{\mathbb{P}^5}(6),$$

and there is a positive rational number $\lambda < 7/6$ such that $(\mathbb{P}^5, \lambda D)$ is strictly log canonical.

Let S be a minimal center in $\mathbb{LCS}(\mathbb{P}^5, \lambda D)$, let Z be the $\bar{G}$-orbit of the subvariety $S \subset \mathbb{P}^5$, and let r be the number of irreducible components of the subvariety Z. We may assume that

$$\mathbb{LCS}\Big(\mathbb{P}^5, \lambda D\Big) = \bigcup_{g \in \bar{G}} \Big\{ g(S) \Big\}$$

by Lemma 2.5. Then $\mathrm{Supp}(Z) = \mathrm{LCS}(\mathbb{P}^5, \lambda D)$. It follows from Lemma 2.2 that

$$g(S) \cap g'(S) \neq \varnothing \iff g(S) = g'(S)$$

for every $g \in \bar{G} \ni g'$. Then $\deg(Z) = r\deg(S)$.

It follows from Remark 3.4 that the subgroup G does not have invariants of degree up to 6, which immediately implies that $\dim(S) \neq 4$ by Remark 4.1.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the log pair $(\mathbb{P}^5, \lambda D)$, and let $\mathcal{L}$ be the log canonical singularities subscheme of the log pair $(\mathbb{P}^5, \lambda D)$. By Theorem 2.1, there is an exact sequence

$$0 \longrightarrow H^0\Big(\mathcal{O}_{\mathbb{P}^5}(n) \otimes \mathcal{I}\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathbb{P}^5}(n)\Big) \longrightarrow H^0\Big(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^5}(n)\Big) \longrightarrow 0$$

for every $n \geqslant 1$. A direct computation shows that $\mathrm{Sym}^n(V)$ is an irreducible representation of the group G for all $n \leqslant 5$. Hence, we see that

$$h^0\Big(\mathcal{O}_{\mathbb{P}^5}(n) \otimes \mathcal{I}\Big) = 0$$

for every $n \in \{1,2,3,4,5\}$. Note that $Z = \mathcal{L}$, because $(\mathbb{P}^5, \lambda D)$ is log canonical. Thus, we have

$$h^0\Big(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}^5}(n)\Big) = h^0\Big(\mathcal{O}_{\mathbb{P}^5}(n)\Big) = \binom{5+n}{n} \tag{4.3}$$

for every $n \in \{1,2,3,4,5\}$. In particular, we see that $r \leqslant 6$, because

$$rh^0\Big(\mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}^5}(1)\Big) = h^0\Big(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}^5}(1)\Big) = 6,$$

which implies that $r = 1$, because $\bar{G}$ has no nontrivial maps to S_r for $2 \leqslant r \leqslant 6$.

Note that it follows from the equality $r = 1$ that $\dim(S) \neq 0$.

Let H be a hyperplane section of the variety $S \subset \mathbb{P}^5$. It follows from Theorem 2.3 that the variety S is normal and has at most rational singularities, and there are an effective $\mathbb{Q}$-divisor B_S

and an ample $\mathbb{Q}$-divisor Δ on the surface S such that $K_S + B_S + \Delta \sim_{\mathbb{Q}} H$, and the log pair (S, B_S) has Kawamata log terminal singularities.

Using the Riemann–Roch theorem and Remark 2.4, we see that $\chi(\mathcal{O}_S(nH))$ is a polynomial in n of degree at most $\dim(S)$ such that

$$\chi\Big(\mathcal{O}_S(nH)\Big) = h^0\Big(\mathcal{O}_S(nH)\Big)$$

for any $n \geqslant 1$. On the other hand, it follows from (4.3) that

$$\chi\Big(\mathcal{O}_S(nH)\Big) = \begin{cases} 6 \text{ if } n = 1, \\ 21 \text{ if } n = 2, \\ 56 \text{ if } n = 3, \\ 126 \text{ if } n = 4, \\ 252 \text{ if } n = 5, \end{cases}$$

which gives an inconsistent system of linear equations on the coefficients of the polynomial $\chi(\mathcal{O}_S(nH))$, since $\dim(S) \leqslant 3$. □

Theorem 4.4. Suppose that $G \cong 6.\mathrm{A}_7$. Then $\mathrm{lct}(\mathbb{P}^5, \bar{G}) \geqslant 7/6$.

Proof. Suppose that $\mathrm{lct}(\mathbb{P}^5, \bar{G}) < 7/6$. Then there is an effective $\bar{G}$-invariant $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^5} \sim \mathcal{O}_{\mathbb{P}^5}(6),$$

and there is a positive rational number $\lambda < 7/6$ such that $(\mathbb{P}^5, \lambda D)$ is strictly log canonical.

Arguing as in the proof of Theorem 4.2, we may assume that

$$\mathbb{LCS}\Big(\mathbb{P}^5, \lambda D\Big) = \bigcup_{g \in \bar{G}} \Big\{g(S)\Big\},$$

where S is a minimal center of log canonical singularities of the log pair $(\mathbb{P}^5, \lambda D)$.

Let Z be the $\bar{G}$-orbit of the subvariety $S \subset \mathbb{P}^5$. Then $\mathrm{LCS}(\mathbb{P}^5, \lambda D) = \mathrm{Supp}(Z)$.

It follows from Remark 3.4 that the subgroup G does not have invariants of degree up to 6, which immediately implies that $\dim(S) \neq 4$ by Remark 4.1.

Let $\mathcal{I}$ be the multiplier ideal sheaf of the log pair $(\mathbb{P}^5, \lambda D)$, and let $\mathcal{L}$ be the log canonical singularities subscheme of the log pair $(\mathbb{P}^5, \lambda D)$. By Theorem 2.1, we have

$$\chi\Big(\mathcal{O}_Z(nH)\Big) = h^0\Big(\mathcal{O}_Z(nH)\Big) = \binom{5+n}{n} - h^0\Big(\mathcal{O}_{\mathbb{P}^5}(n) \otimes \mathcal{I}\Big), \tag{4.5}$$

for every $n \geqslant 1$, because $Z = \mathcal{L}$. Put $q_n = h^0(\mathcal{O}_{\mathbb{P}^5}(n) \otimes \mathcal{I})$ for every $n \geqslant 1$. Then

$$q_1 = q_2 = 0,$$

because V and $\mathrm{Sym}^2(V)$ are irreducible representations of the group G. Hence

$$h^0\Big(\mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}^5}(n)\Big) = h^0\Big(\mathcal{O}_{\mathbb{P}^5}(n)\Big) = \binom{5+n}{n} \tag{4.6}$$

for $n \in \{1, 2\}$ by (4.5). In particular, we see that $r \leqslant 6$, because

$$rh^0\Big(\mathcal{O}_S \otimes \mathcal{O}_{\mathbb{P}^5}(1)\Big) = h^0\Big(\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}^5}(1)\Big) = 6,$$

which implies that $r = 1$ and $Z = S$, because $\bar{G}$ has no nontrivial maps to S_r for $2 \leqslant r \leqslant 6$.

Note that it follows from the equality $r = 1$ that $\dim(S) \neq 0$.

Similarly, we see that $q_3 \in \{0, 20, 36\}$, because

$$\mathrm{Sym}^3(V) = T_{36} \oplus T_{20},$$

where T_i is an irreducible representation of the group G of dimension i. Thus, we have

$$h^0\Big(\mathcal{O}_S\otimes\mathcal{O}_{\mathbb{P}^5}\big(3\big)\Big)=h^0\Big(\mathcal{O}_{\mathbb{P}^5}\big(3\big)\Big)-h^0\Big(\mathcal{O}_{\mathbb{P}^5}\big(3\big)\otimes\mathcal{I}\Big)=56-h^0\Big(\mathcal{O}_{\mathbb{P}^5}\big(3\big)\otimes\mathcal{I}\Big)\in\big\{20,36,56\big\} \tag{4.7}$$

by (4.5). Moreover, one has

$$\mathrm{Sym}^4\big(V\big)=U_6\oplus U_{15}\oplus\hat{U}_{15}\oplus U_{21}^{\oplus 2}\oplus U_{24}\oplus\hat{U}_{24},$$

where U_i and $\hat{U}_i$ are irreducible representations of the group G of dimension i. In particular,

$$h^0\Big(\mathcal{O}_S\otimes\mathcal{O}_{\mathbb{P}^5}\big(4\big)\Big)=126-h^0\Big(\mathcal{O}_{\mathbb{P}^5}\big(4\big)\otimes\mathcal{I}\Big)\not\in\big\{106,114,\ldots,119,121,\ldots,125\big\} \tag{4.8}$$

by (4.5). Finally, one has

$$\mathrm{Sym}^5\big(V\big)=W_{11}^{\oplus 2}\oplus W_{24}^{\oplus 2}\oplus\hat{W}_{24}^{\oplus 2}\oplus W_{36}^{\oplus 4},$$

where W_i and $\hat{W}_i$ are irreducible representations of the group G of dimension i. By (4.5), we have

$$h^0\Big(\mathcal{O}_S\otimes\mathcal{O}_{\mathbb{P}^5}\big(5\big)\Big)=252-h^0\Big(\mathcal{O}_{\mathbb{P}^5}\big(5\big)\otimes\mathcal{I}\Big)\not\in\big\{66,171,179\big\}. \tag{4.9}$$

Let H be a hyperplane section of the variety $S\subset\mathbb{P}^5$. It follows from Theorem 2.3 that the variety S is normal and has at most rational singularities, and there are an effective $\mathbb{Q}$-divisor B_S and an ample $\mathbb{Q}$-divisor Δ on the surface S such that $K_S+B_S+\Delta\sim_{\mathbb{Q}}H$, and the log pair (S,B_S) has Kawamata log terminal singularities.

Suppose that $\dim(S)=1$. Then S is a smooth curve of genus g such that

$$\deg\big(H\big)=\deg\big(S\big)>2g-2,$$

and $\deg(Z)\leqslant 15$ by Lemma 2.6. By the Riemann–Roch theorem, we get

$$h^0\Big(\mathcal{O}_S\big(nH\big)\Big)=n\deg\big(S\big)-g+1$$

for every $n\geqslant 1$ (see Remark 2.4). Using (4.6), we see that

$$\begin{cases} 6=\deg\big(S\big)-g+1,\\ 21=2\deg\big(S\big)-g+1,\end{cases}$$

which implies that $\deg(S)=15$ and $g=10$. Using (4.6) again one obtains

$$5\deg\big(S\big)-g+1=66,$$

which is impossible by (4.9).

Suppose that $\dim(S)=2$. Using the Riemann–Roch theorem and Remark 2.4, we have

$$h^0\Big(\mathcal{O}_S\big(nH\big)\Big)=\chi\Big(\mathcal{O}_S\big(nH\big)\Big)=\frac{n^2}{2}\Big(H\cdot H\Big)-\frac{n}{2}\Big(H\cdot K_S\Big)+\chi\big(\mathcal{O}_S\big) \tag{4.10}$$

for any $n\geqslant 1$. Thus, using (4.6) and (4.7), we see that

$$\Big(\deg\big(S\big),H\cdot K_S,\chi\big(\mathcal{O}_S\big)\Big)\in\Big\{\big(5,-5,6\big),\big(15,5,-4\big)\Big\},$$

because $H\cdot H=\deg(S)>0$. If $(\deg(S),H\cdot K_S,\chi(\mathcal{O}_S))=(15,5,-4)$, then

$$h^0\Big(\mathcal{O}_S\big(4H\big)\Big)=8\Big(H\cdot H\Big)-2\Big(H\cdot K_S\Big)+\chi\big(\mathcal{O}_S\big)=106,$$

which is impossible by (4.8). If $(\deg(S),H\cdot K_S,\chi(\mathcal{O}_S))=(15,5,6)$, then

$$h^0\Big(\mathcal{O}_S\big(4H\big)\Big)=116,$$

which is again impossible by (4.8).

We see that $\dim(S)=3$. Then $H\cdot H\cdot H=\deg(S)\geqslant\mathrm{codim}(S)+1=3$, since V is an irreducible representation of the group G.

Let H' be another general hyperplane section of $S \subset \mathbb{P}^5$. Put $C = H \cap H'$. Then

$$-2 \leqslant 2g\big(C\big) - 2 = H \cdot H \cdot K_S + 2\Big(H \cdot H \cdot H\Big),$$

where $g(C)$ is the genus of the curve C. Thus, we see that

$$H \cdot H \cdot K_S \geqslant -2 - 2\mathrm{deg}\big(S\big). \tag{4.11}$$

By the Riemann–Roch theorem and Remark 2.4, there is $\gamma \in \mathbb{Z}$ such that

$$h^0\Big(\mathcal{O}_S\big(nH\big)\Big) = \chi\Big(\mathcal{O}_S\big(nH\big)\Big) = \frac{n^3}{6}\Big(H \cdot H \cdot H\Big) + \frac{n^2}{4}\Big(H \cdot H \cdot K_S\Big) + \frac{n}{12}\gamma + \chi\big(\mathcal{O}_S\big)$$

for any $n \geqslant 1$. Put $h_n = h^0(\mathcal{O}_S(nH))$. Then

$$h_4 - 3h_3 + 3h_2 - h_1 = H \cdot H \cdot H, \tag{4.12}$$

and

$$h_3 - 2h_2 + h_1 = 2\Big(H \cdot H \cdot H\Big) + \frac{1}{2}\Big(H \cdot H \cdot K_S\Big), \tag{4.13}$$

which implies after applying (4.11)

$$h_4 \leqslant 2h_1 - 5h_2 + 4h_3 + 1. \tag{4.14}$$

Since $H \cdot H \cdot H \geqslant 3$, the equality (4.12) also implies

$$h_4 \geqslant 3 + h_1 - 3h_2 + 3h_3. \tag{4.15}$$

Recall that $h_1 = 6$, $h_2 = 20$, $h_3 \in \{20, 36, 56\}$.

If $h_3 = 20$, then (4.14) implies that $h_4 \leqslant -12$, which is a contradiction.

If $h_3 = 36$, then (4.14) and (4.15) imply that $52 \geqslant h_4 \geqslant 54$, which is a contradiction.

We see that $h_3 = 56$. Then (4.15) implies that $h_4 \geqslant 114$, so that

$$h_4 \in \big\{120, 126\big\}$$

by (4.8). If $h_4 = 120$, then (4.12) implies that $H \cdot H \cdot H = 9$. Note that

$$H \cdot H \cdot H = h_5 - 3h_4 + 3h_3 - h_2, \tag{4.16}$$

and hence $h_5 = 171$, which is impossible by (4.9). Thus, we see that $h_4 = 126$.

It follows from (4.12) and (4.16) that $H \cdot H \cdot H = 15$ and $h_5 = 179$, which is impossible by (4.9). $\square$

5. Seven-dimensional singularities

Let G be a finite subgroup in $\mathrm{SL}_7(\mathbb{C})$, and let $\phi\colon \mathrm{SL}_7(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^6)$ be the natural projection. Put $\bar{G} = \phi(G)$. We say that the subgroup G is the lift of the subgroup $\bar{G}$ to $\mathrm{SL}_7(\mathbb{C})$.

Theorem 5.1 ([32, Theorem 4.1], [33, Theorem I])**.** Suppose that the subgroup G is quasiprimitive. Then there is a lift of the subgroup $\bar{G}$ to $\mathrm{SL}_7(\mathbb{C})$ that is contained in the following list:

- (I) a subgroup of the subgroup $G_7 \subset \mathrm{SL}_7(\mathbb{C})$ such that

  $$G_7 = \mathrm{Norm}_{\mathrm{SL}_7(\mathbb{C})}(\mathbb{H}_7) \cong \mathbb{H}_7 \rtimes \mathrm{SL}_2\Big(\mathbb{F}_7\Big),$$

  where $\mathbb{H}_7$ is the Heisenberg group of order 7^3, and the corresponding seven-dimensional representation of the group $\mathbb{H}_7$ is any of its 7-dimensional irreducible representations,
- (II) $\mathrm{PSL}_2(\mathbb{F}_{13})$,
- (III) (i) $\mathrm{PSL}_2(\mathbb{F}_8)$,
  - (ii) an extension of the subgroup described in III(i) by an automorphism of order 3,
- (IV) A_8 or S_8,
- (V) (i) $\mathrm{PSL}_2(\mathbb{F}_7)$,
  - (ii) $\mathrm{PGL}_2(\mathbb{F}_7)$,

(VI) (i) $\mathrm{PSU}_3(\mathbb{F}_3)$,
(ii) an extension of the subgroup described in VI(i) by an automorphism of order 2,
(VII) $\mathrm{Sp}_6(\mathbb{F}_2)$.

Remark 5.2. Up to conjugation there are two primitive subgroups in $\mathrm{SL}_7(\mathbb{C})$ that are isomorphic to $\mathrm{PSL}_2(\mathbb{F}_8)$: one is conjugate to a subgroup in $\mathrm{SL}_7(\mathbb{Q})$, and another is conjugate to a subgroup in $\mathrm{SL}_7(\mathbb{Q}(\xi_9))$, where ξ_9 is a primitive root of unity of degree 9. Similarly, up to conjugation there are two primitive subgroups in $\mathrm{SL}_7(\mathbb{C})$ that are isomorphic to $\mathrm{PSU}_3(\mathbb{F}_3)$: one is conjugate to a subgroup in $\mathrm{SL}_7(\mathbb{Q})$, another is conjugate to a subgroup in $\mathrm{SL}_7(\mathbb{Q}(\sqrt{-1}))$. The detailed information on the corresponding representations may be found in [10].

The main purpose of this section is to prove the following result.

Theorem 5.3. Suppose that G is quasiprimitive. Then either G has a semi-invariant of degree at most 7, or G is a subgroup of the subgroup $G_7 \subset \mathrm{SL}_7(\mathbb{C})$ (see Theorem 5.1).

Proof. Recall that changing a lift of $\bar{G}$ to $\mathrm{SL}_7(\mathbb{C})$ does not change the degrees of semi-invariants, which implies that we may assume that G is one of the groups listed in Theorem 5.1.

Note that the groups of type I have a unique lift to $\mathrm{SL}_7(\mathbb{C})$.

By Lemma 2.7, we may assume that G is not conjugate to a subgroup in $\mathrm{SL}_7(\mathbb{Q})$, which implies that the subgroup G is not of type IV, V(i) or VII (see [10]).

If the subgroup G is of type II, III(i) or VI(i) (cf. Remark 5.2), then the minimal degree d_{min} of the invariants of the subgroup G is given in the following table:

G	$\mathrm{PSL}_2(\mathbb{F}_{13})$	$\mathrm{PSL}_2(\mathbb{F}_8)$	$\mathrm{PSU}_3(\mathbb{F}_3)$
Type	II	III(i)	VI(i)
d_{min}	2	2	3

If the subgroup G is a subgroup of type III(ii), then it has a normal subgroup of index 3 isomorphic to $\mathrm{PSL}_2(\mathbb{F}_8)$, which implies that G has a semi-invariant of degree 2 by Lemma 2.8.

Arguing as in the case of a subgroup of type III(ii), we see that the subgroup G has a semi-invariant of degree at most 3 if G is a quasiprimitive subgroup of type V(ii) or VI(ii). □

Proof of Theorem 1.16. By [22, Lemma 2.2.1(xi)] (see also [21, §1]), the group $G_7 \subset \mathrm{SL}_7(\mathbb{C})$ has an invariant of degree 7. Thus, Theorem 5.3 implies Theorem 1.16. □

Appendix A. Alpha-invariant

Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity (see [13, Definition 3.5]), and let $\pi\colon W \to V$ be a birational morphism such that

- the exceptional locus of π consists of one irreducible divisor $E \subset W$ such that $O \in \pi(E)$,
- the log pair (W, E) has purely log terminal singularities (see [13, Definition 3.5]),
- the divisor $-E$ is a π-ample $\mathbb{Q}$-Cartier divisor.

Theorem A.1. The birational morphism $\pi\colon W \to V$ does exist.

Proof. The required assertion follows from [23, Proposition 2.9], [14, Theorem 1.5] and [1]. □

The existence of π is obvious if $(V \ni O)$ is a quotient singularity (see [7, Remark 3.15]) or an isolated quasi-homogeneous hypersurface singularity.

Definition A.2 ([23, Definition 2.1])**.** We say that π is a plt blow up of the germ $(V \ni O)$.

Definition A.3 ([23, Definition 4.1])**.** We say that $(V \ni O)$ is weakly-exceptional if π is unique.

The goal of this appendix is to define an invariant $\alpha(V \ni O) \in \mathbb{R}$ of the singularity $(V \ni O)$, which is a local analogue of the α-invariant introduced in [30] and [31].

Lemma A.4 (see [23, Theorem 4.9])**.** If $(V \ni O)$ is exceptional, then $\pi(E) = O$.

Lemma A.5 ([14, Corollary 1.7],[1])**.** If $(V \ni O)$ is weakly-exceptional, then $\pi(E) = O$.

If $\pi(E) \neq O$, then we put $\alpha(V \ni O) = 0$. Suppose, in addition, that $\pi(E) = O$.

Denote by $R_1, \ldots, R_s$ the irreducible components of the locus $\mathrm{Sing}(W)$ such that $\dim(R_i) = \dim(V) - 2$ and $R_i \subset E$ for every $i \in \{1, \ldots, s\}$. Put

$$\Delta = \sum_{i=1}^{s} \frac{m_i - 1}{m_i} R_i,$$

where m_i is the smallest positive integer such that $m_i E$ is Cartier at a general point of the subvariety $R_i \subset E$. (One has $\Delta = \mathrm{Diff}_E(0)$ in the notation of the paper [23].)

Lemma A.6 ([13, Theorem 7.5])**.** The variety E is normal, and the log pair $(E, \mathrm{Diff}_E(0))$ is Kawamata log terminal.

The log pair (E, Δ) is a log Fano variety, i.e. the divisor $-(K_E + \Delta)$ is ample. Indeed, the divisor $-E$ is π-ample, and

$$K_E + \Delta \sim_{\mathbb{Q}} \Big(K_W + E\Big)\Big|_E \sim_{\mathbb{Q}} \Big(\pi^*\big(K_V\big) + \big(1 + a\big)E\Big)\Big|_E.$$

Moreover, one has $a > -1$, because V has Kawamata log terminal singularities. Put

$$\mathrm{lct}\Big(E, \Delta\Big) = \sup\left\{ \lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (E, \Delta + \lambda D) \text{ is log canonical} \\ \text{for any effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -\Big(K_E + \Delta\Big) \end{array} \right\}.$$

Theorem A.7 ([14, Theorem 2.1])**.** The singularity $(V \ni O)$ is weakly-exceptional if and only if the inequality $\mathrm{lct}(E, \Delta) \geqslant 1$ holds.

Note that the real number $\mathrm{lct}(E, \Delta)$ is an algebraic counter-part of the so-called α-invariant introduced in [30] and [31] (cf. [6, Theorem A.3]). Put

$$\alpha\big(V \ni O\big) = \begin{cases} \mathrm{lct}\big(E, \Delta\big) \text{ if } \mathrm{lct}\big(E, \Delta\big) \geqslant 1, \\ 0 \text{ if } \mathrm{lct}\big(E, \Delta\big) < 1. \end{cases}$$

Definition A.8. We say that $\alpha(V \ni O)$ is the alpha-invariant of the singularity $(V \ni O)$.

Note that $\alpha(V \ni O) \neq 0 \iff \alpha(V \ni O) \geqslant 1 \iff (V \ni O)$ is weakly-exceptional.

Example A.9 ([5, Lemma 5.2])**.** Suppose that $(V \ni O)$ is an isolated quasi-homogeneous hypersurface singularity

$$z^2t + yt^2 + xy^4 + x^8z = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $O \in V$ is given by $x = y = z = t = 0$. Then $\alpha(V \ni O) = 33/4$.

Theorem A.10 ([23, Theorem 4.9])**.** The the singularity $(V \ni O)$ is exceptional if and only if for every effective $\mathbb{Q}$-divisor D on the variety E such that $D \sim_{\mathbb{Q}} -(K_E + \Delta)$ the log pair $(E, \Delta + D)$ has Kawamata log terminal singularities.

Corollary A.11. The singularity $(V \ni O)$ is exceptional if $\alpha(V \ni O) > 1$.

Corollary A.12. If $(V \ni O)$ is exceptional, then $(V \ni O)$ is weakly-exceptional.

Let G be a finite subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$, where $n \geqslant 1$. Put

$$\bar{G} = \phi\big(G\big) \subset \mathrm{Aut}\big(\mathbb{P}^n\big) \cong \mathrm{PGL}_{n+1}\Big(\mathbb{C}\Big),$$

where $\phi\colon \mathrm{GL}_{n+1}(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^n)$ is the natural projection. Put

$$\mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) = \sup\left\{\lambda \in \mathbb{Q}\ \left|\ \begin{array}{l}\text{the log pair } (\mathbb{P}^n, \lambda D) \text{ has log canonical singularities}\\ \text{for every } \bar{G}\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n}\end{array}\right.\right\} \in \mathbb{R}.$$

Lemma A.13 ([7, Remark 3.2])**.** Suppose that G does not contain reflections. Then

$$\alpha\big(V \ni O\big) = \begin{cases} \mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) \text{ if } \mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) \geqslant 1,\\ 0 \text{ if } \mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) < 1.\end{cases}$$

We see that the number $\alpha\big(V \ni O\big)$ measures how exceptional $(V \ni O)$ is (cf. Remark 1.13).

Example A.14. Let $G \subset \mathrm{PSL}_3(\mathbb{C})$. Then

$$\mathrm{lct}\Big(\mathbb{P}^n, \bar{G}\Big) = \begin{cases} 4/3 \text{ if } \bar{G} \cong \mathrm{PSL}_2\Big(\mathbb{F}_7\Big),\\ 2 \text{ if } \bar{G} \cong \mathrm{A}_6,\end{cases}$$

by [4, Examples 1.9 and 6.5].

For a fixed $n \in \mathbb{Z}_{>0}$, the number $\mathrm{lct}(\mathbb{P}^n, \bar{G})$ is bounded by Theorem 1.11, because there exist only finitely many primitive finite subgroups in $\mathrm{SL}_{n+1}(\mathbb{C})$ up to conjugation (see [9]).

Theorem A.15 ([7, Theorem 1.24])**.** The inequality $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leqslant 4(n+1)$ holds for every $n \geqslant 1$.

In fact, we expect the following to be true (cf. [29]).

Conjecture A.16. There is $\alpha \in \mathbb{R}$ such that $\mathrm{lct}(\mathbb{P}^n, \bar{G}) \leqslant \alpha$ for any $\bar{G} \subset \mathrm{Aut}(\mathbb{P}^n)$ and $n \geqslant 1$.

One can try to tackle Conjecture A.16 using the classification of finite simple groups.

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